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3+1 dimensional Yang-Mills theory as a local theory of evolution of metrics on 3 manifolds.

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Abstract

An explicit canonical transformation is constructed to relate the physical subspace of Yang-Mills theory to the phase space of the ADM variables of general relativity. This maps 3+1 dimensional Yang-Mills theory to local evolution of metrics on 3 manifolds.

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The question of whether the dynamics of Yang-Mills theory can be completely captured in terms of gauge invariant quantitites has been raised many times. This is important especially because of confinement in QCD. One approach has been to rewrite the theory as dynamics of the loop variables. These Wilson loops are non-local variables and they also form an over-complete set. The possibility of using the gauge invariant combination $\vec{E}^i \cdot \vec{E}^j$ of the non-Abelian electric field has also been explored [1] [2] [3]. In another approach, the gauge invariant variables $\vec{B}_i[A] \cdot \vec{B}_j[A]$ have also been considered [4] [5]. Analogy to gravity yields a nice geometric interpretation for 2+1 dimensional Yang-Mills theory [6]. Such an approach for 3+1 dimensions has been attempted in [7]

In this article we use certain techniques motivated by the Ashtekar formulation of gravity [8]. We map the physical phase space of Yang-Mills theory to the phase space of the ADM variables of general relativity by an explicit canonical transformation. To do this we augment the ADM variables (g_{ij}, π^{ij}) by a set of auxiliary variables (θ_a, χ^a) to match in number, the variables (A^a_i, E^{ia}) of the extended phase space of Yang-Mills theory. It turns out that the non-Abelian Gauss law simply becomes the constraint $\chi^a = 0$. Therefore the physical subspace of the phase space of the Yang-Mills theory is exactly mapped to the phase space of the ADM variables and the dynamics can be rewritten as a local theory of evolution of metrics on 3-manifolds.

We use the language of functional integrals, but every step below may be interpreted in terms of dynamics of the classical theory. We begin with the Euclidean partition function

$$Z = \int \mathcal{D}A^a_\mu \; exp\{-\frac{1}{4g^2}\int (\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + \vec{A}_\mu \times \vec{A}_\nu)^2\}. \tag{1}$$

Introducing an auxiliary field E^{ia}, and integrating over A^a_0, we get

$$Z = \int \mathcal{D}A^a_i \mathcal{D}E^{ia} \; \delta(D_i[A]E^i) \; exp\{\int(-\mathcal{H} + i\vec{E}^i \cdot \partial_0 \vec{A}_i)\} \tag{2}$$

where

$$\mathcal{H} = \frac{1}{2}(g^2 E^2 + \frac{1}{g^2}B^2), \tag{3}$$

is the hamiltonian density, and

$$\vec{B}^i[A] = \frac{1}{2}\epsilon^{ijk}(\partial_j \vec{A}_k - \partial_k \vec{A}_j + \vec{A}_j \times \vec{A}_k) \tag{4}$$

is the non-Abelian magnetic field. Using the Feynman time slicing procedure, it is also clear that A_i, E^i are the conjugate variables of the phase space. There are also three first class constraints, the non-Abelian Gauss law :

$$D_i[A]\vec{E}^i = 0. \tag{5}$$

Motivated by the Ashtekar variables, we define a driebein e by

$$\vec{E}^i = \frac{1}{2}\epsilon^{ijk}\vec{e}_j \times \vec{e}_k \tag{6}$$

Assuming $||E|| = ||e||^2$ is nonzero, we can invert (6) to get $e_i^a = ||E||^{\frac{1}{2}}(E^{-1})_i^a$. Define $\bar{A}[E]$ as the connection one form which is torsion-free with respect to the driebein $\vec{e}_i$. We have

$$\epsilon_{ijk}(\partial_j \vec{e}_k + \vec{\bar{A}}_j(E) \times \vec{e}_k) = 0 \tag{7}$$

Therefore

$$D_i[\bar{A}(e)]\vec{E}^i = 0 \tag{8}$$

is identically valid. Hence we may replace Gauss law (5) by

$$\vec{a}_i \times \vec{E}^i = 0 \tag{9}$$

where $a_i = A_i - \bar{A}_i(E)$ transforms homogeneously under gauge transformation.

We now observe that the change of variables from $\{A_i, E^i\}$ to $\{a_i, E^i\}$ is a canonical transformation. Consider the generating function

$$S[a_i^a, E^{ia}] = \int d^3x \; a_i^a E^{ia} \; + \; \bar{S}[e] \tag{10}$$

where

$$\bar{S}[e] = \frac{1}{2}\int d^3x \, \epsilon^{ijk}\vec{e}_i \cdot \partial_j \vec{e}_k \tag{11}$$

The momentum conjugate to the new coordinate a_i^a is $\frac{\delta S}{\delta a_i^a} = E^{ia}$, same as for A_i^a. The relation between the old and the new coordinates is $A_i^a = \frac{\delta S}{\delta E^{ia}}$ so that $A_i^a - a_i^a = \frac{\delta \bar{S}}{\delta E^{ia}}$. Now

$$\frac{\delta \bar{S}[e]}{\delta e_l^d} = \epsilon^{ljm}\partial_j e_m^d = -\epsilon^{ljm}\epsilon^{def}\bar{A}_j^e[E]e_m^f. \tag{12}$$

Therefore we have,

$$\frac{\delta \bar{S}}{\delta E^{ia}} = \frac{\delta e_l^d}{\delta E^{ia}}\frac{\delta \bar{S}[e]}{\delta e_l^d} = \bar{A}_i^e[E] \tag{13}$$

since

$$\frac{\delta e_l^d}{\delta E^{ia}} = \frac{1}{||e||}(\frac{1}{2}e_l^d e_i^a - e_i^d e_l^a). \tag{14}$$

We now show that there exists a canonical transformation from the phase space (a, E) with the constraint (9) to the phase space of the ADM variables (π^{ij}, g_{ij}). (The first class constraints involving the ADM variables, related to space-time translations is not relevant for the present context of Yang-Mills theory.) As the variables (a, E) are more in number than (π, g), we first augment the latter set by a canonically conjugate set $\{\chi^a, \theta_a\}$. Here

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j, \tag{15}$$

and to define θ_a we consider the Polar decomposition of e_i^a:

$$e_i^a = e_{ij} O^{ja} \tag{16}$$

into a symmetric matrix e_{ij} and an orthogonal matrix O^{ja}. θ^a is the lie algebra element corresponding to O^{ja}.

$$O = exp(i\theta_a \frac{T^a}{2}). \tag{17}$$

The equations (15-17) give us the relation between the old momenta and the new coordinates. We relate the momenta via a generator of the canonical transformation

$$S(\pi, \chi, E) = \int \Big((\vec{e}_i \cdot \vec{e}_j)\pi^{ij} + \theta_a(e)\chi^a \Big) \tag{18}$$

This gives us $g_{ij} \equiv \frac{\delta S}{\delta \pi^{ij}} = \vec{e}_i \cdot \vec{e}_j$ as we want. Among the other quantitites, θ_a is given by

$$\theta_a \equiv \frac{\delta S}{\delta \chi^a} = \theta_a[e]. \tag{19}$$

Also we have

$$a_i^a \equiv \frac{\delta S}{\delta E^{ia}} \tag{20}$$

To express a_i^a in terms of the other variables, we take the variation of S with respect to e_i^a. We get

$$\frac{\delta S}{\delta e_i^a} = 2\pi^{ij} e_j^a + \frac{1}{2}\chi^b \epsilon_{ijk} M^{-1}[\theta]_c^b ((e - \mathbf{1}\, tr\, e)^{-1})_{kl}\, O^{ja} O^{lc}. \tag{21}$$

where M is defined in the appendix. Now note that

$$\frac{\delta S}{\delta e_i^a} = \frac{\delta E^{bj}}{\delta e_i^a} a_j^b. \tag{22}$$

This explicitly relates a to the other variables. We now show that χ is related to the Gauss law. Contracting $\frac{\delta S}{\delta e_i^a}$ by e_i^c, we get from (22)

$$e_i^c \frac{\delta S}{\delta e_i^a} = -E^{ia} a_i^c + E^{ib} a_{ib} \delta_{ac} \tag{23}$$

For the part antisymmetric in (a, c), we get from (21),

$$(\vec{a}_i \times \vec{E}^i)^a = \frac{1}{2}(M^{-1})_b^a[\theta]\chi^b \tag{24}$$

Thus the canonical momentum π^{ij} drops out in the Gauss law equation (5). *With the new variables, the Gauss law is implemented by simply setting* $\chi = 0$.

The partition function (2) in the new variables is ,

$$Z = \int \mathcal{D}g_{ij}\, \mathcal{D}\pi^{ij}\, \mathcal{D}\theta^a\, \mathcal{D}\chi^a \delta((M^{-1})_b^a[\theta]\chi^b) exp \int (-\mathcal{H}' + i\pi^{ij}\partial_0 g_{ij} + i\chi^a \partial_0 \theta^a). \tag{25}$$

θ^a represents the gauge degrees of freedom. We may adopt the Faddeev-Popov procedure to choose $\theta^a = 0$. In this case $M^a_b[\theta] = \delta^a_b$, and

$$Z = \int \mathcal{D}g_{ij}\, \mathcal{D}\pi^{ij}\, exp \int (-\mathcal{H}^{'}[g,\pi] + i\pi^{ij}\partial_0 g_{ij}) \tag{26}$$

Thus the functional integral is rewritten in terms of the conjugate variables (g_{ij}, π^{ij}) which are gauge invariant. The new Hamiltonian $\mathcal{H}^{'}$ is obtained from (3), by the replacements

$$E^{ia} \rightarrow \frac{1}{2}\epsilon^{ijk}\epsilon^{abc}e_{jb}e_{kc} \tag{27}$$

$$A^a_i \rightarrow \bar{A}^a_i[E] + a^a_i \tag{28}$$

$$a^a_i \rightarrow \frac{1}{||e||}(\pi^{jk}g_{jk}e_{ia} - 2\pi^{jk}g_{ik}e_{ja}) \tag{29}$$

where e_{ia} is regarded as the symmetric square root of g_{ij}. Thus the $(\vec{E}^i)^2$ term in the Hamiltonian becomes $\frac{g^{ii}}{||g||}$, while

$$B_i[A] = B_i[\bar{A}[E]] + \epsilon_{ijk}D_j[\bar{A}[E]]a_k + \frac{1}{2}\epsilon_{ijk}(a_j \times a_k) \tag{30}$$

corrsponding to an expansion about a ‘background’ gauge field $\bar{A}[E]$. The $(\vec{B}_i[A])^2$ can be completely written as *local* expressions in g_{ij} and π^{ij}. For example

$$B^a_i[\bar{A}[E]] = \frac{1}{4||e||}\epsilon_{ijk}\,\epsilon_{lmn}R_{jl}\,g_{km}\,e^a_n \tag{31}$$

$$\epsilon_{ijk}(a_j \times a_k)\cdot\epsilon_{imn}(a_m \times a_n) = \vec{a}_j\cdot\vec{a}_m\,\vec{a}_j\cdot\vec{a}_m - (\vec{a}_j\cdot\vec{a}_j)^2 \tag{32}$$

Similarly

$$(D_i[\bar{A}[E]]e_j)^a = \Gamma^k_{ij}[g]e^a_k \tag{33}$$

where Γ^k_{ij} is the affine connection corresponding to the metric g_{ij}. $D_k[\bar{A}[E]]$ can be replaced by the covariant derivative corresponding to the affine connection $\Gamma^k_{ij}[g]$ when acting on g_{ij} or π^{ij}. This way, $(\vec{B}_i[A])^2$ can be written as a local expression in g_{ij} and π^{ij}.

We have thus mapped the physical phase space of Yang-Mills theory onto the phase space of the ADM variables and the dynamics is now a local evolution of the metrics on 3-manifolds. This completes the program envisaged in [1].

The canonical transformation constructed here can be used to map all the ADM constraints of general relativity to certain constraints on the Yang-Mills phase space without requiring complexification of the gauge field. In fact Barbero’s constraints [9] are reproduced.

Here we have related the gauge invariant combination $\vec{E}^i\cdot\vec{E}^j$ to a metric on a 3-manifold. It is also possible to construct a metric from the vector potential A^a_i and use it to rewrite the Yang-Mills dynamics. These two approaches are dual of each other. This will be addressed elsewhere.

A. Appendix

To evaluate $\frac{\delta\theta^b[e]}{\delta e_i^a}$, note that

$$e_i^a + \delta e_i^a = (e_{ij} + \delta e_{ij}) exp(i(\theta[e] + \delta\theta[e])^b \frac{T^b}{2}). \tag{34}$$

This gives

$$\delta e_i^a \, O^{ka} = \delta e_{ik} + \epsilon^{dae} e_{ij} O^{jd} O^{ka} \bar{\delta\theta}^e \tag{35}$$

where $\bar{\delta\theta}^e = M_e^f[e]\delta\theta^e$ and the matrix M_e^f is given by

$$O^T(\theta)O(\theta + \delta\theta) \approx 1 + T^a M_b^a(\theta)\delta\theta^b. \tag{36}$$

Taking the antisymmetric part of (35) we can solve for $\frac{\delta\theta^b[e]}{\delta e_i^a}$. We get

$$\frac{\delta\theta^b[e]}{\delta e_i^a} = -\epsilon_{ijk} M^{-1}[\theta]_c^b((e - \mathbf{1}\, tr\, e)^{-1})_{kl} \, O^{ja} O^{lc}. \tag{37}$$

When two or all eigenvalues of the symmetric matrix e_{ij} are degenerate, so are those of $(e - \mathbf{1}\, tr\, e)$. In this case, the variables $\theta[e]$ equation (34) are ill defined and more care is required to define the new variables.

REFERENCES

[1] R.Anishetty, Phys. Rev. D 44, (1991) 1895.
[2] F.A.Lunev, Phys. Lett. B 295, (1992) 99.
[3] A. M. Khvedelidze and H.-P. Pavel, Phys. Rev. D 59, (1999) 105017.
[4] Peter E. Haagensen and Kenneth Johnson, Nucl. Phys. B 439, (1995) 597.
[5] Peter E. Haagensen, Kenneth Johnson and C.S.Lam, Nucl.Phys. B 477, (1996) 273.
[6] Ramesh Anishetty, Pushan Majumdar and H.S.Sharatchandra, Phys. Lett. B 478, (2000) 373
[7] F.A.Lunev, Modern Physics Letters A Vol 9 No. 25, (1994) 2281.
[8] A.Ashtekar, Phys. Rev. Lett. 57, (1986) 2244.
[9] J.F.Barbero G. Phys.Rev. D 49, (1994) 6935.